

ISOMETRY TYPES OF FRAME BUNDLES

WOUTER VAN LIMBEEK

ABSTRACT. We consider the orthonormal frame bundle $F(M)$ of a Riemannian manifold M . A construction of Sasaki defines a canonical Riemannian metric on $F(M)$. We prove that for two closed Riemannian n -manifolds M and N , the frame bundles $F(M)$ and $F(N)$ are isometric if and only if M and N are isometric, except possibly in dimensions 3, 4, and 8. This answers a question of Benson Farb except in dimensions 3, 4, and 8.

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1. INTRODUCTION

Let M be a Riemannian manifold, and let $X := F(M)$ be the orthonormal frame bundle of M . The Riemannian structure on M induces in a canonical way a Riemannian metric on $F(M)$, known as the Sasaki metric, defined as follows. Consider the natural projection $\pi : F(M) \rightarrow M$. Each of the fibers of p is naturally equipped with a free and transitive $SO(n)$ -action, so that this fiber carries an $SO(n)$ -bi-invariant metric g_V . The metric g_V is determined uniquely up to scaling. Further, the Levi-Civita connection on the tangent bundle $TM \rightarrow M$ induces a horizontal subbundle of TM . This in turn induces a horizontal subbundle \mathcal{H} of $TF(M)$. We can pull back the metric on M along π to get a metric g_H on \mathcal{H} . The *Sasaki metric* on $F(M)$ is defined to be $g_S := g_V \oplus g_H$.

Note that g_S is determined uniquely up to scaling of g_V , and hence determined uniquely after fixing a bi-invariant metric on $SO(n)$. Sasaki metrics have been defined and studied first by Sasaki [Sas58, Sas62], and further by O'Neill [O'N66] and Takagi-Yawata [TY91, TY94]. These works have determined many natural properties of Sasaki metrics and connections between the geometry of M and $F(M)$. The following natural question then arises, which was to my knowledge first posed by Benson Farb.

Question 1.1. Let M, N be Riemannian manifolds. If $F(M)$ is isometric to $F(N)$ (with respect to Sasaki metrics on each), is M isometric to N ?

The purpose of this paper is to answer Question 1.1 except when $\dim M = 3, 4$ or 8. The question is a bit subtle, for it is not true in general that an isometry of $F(M)$ preserves the fibers of $F(M) \rightarrow M$. For example, if M is a constant curvature sphere S^n then $F(M)$ is diffeomorphic to $SO(n+1)$. There is a unique Sasaki metric that is isometric to the bi-invariant metric on $SO(n+1)$, but of course there are many isometries of $SO(n+1)$ that do not preserve the fibers of $SO(n+1) \rightarrow S^n$.

As it turns out, manifolds with constant positive curvature are the only Riemannian manifolds whose orthonormal frame bundles admit Killing fields that do not preserve the fibers, as follows from a theorem of Takagi-Yawata [TY91]. However, more examples of non-fiber-preserving isometries appear if we consider isometries that are not induced by Killing fields, as the following example shows.

Example 1.2. Let M be a flat 2-torus obtained as the quotient of \mathbb{R}^2 by the subgroup generated by translations by $(l_1, 0)$ and $(0, l_2)$ for some $l_1, l_2 > 0$. Further fix $l_3 > 0$ and equip $F(M)$ with the Sasaki metric associated to the scalar l_3 . It is easy to see $F(M)$ is the flat 3-torus obtained as the quotient of \mathbb{R}^3 by the subgroup generated by $(l_1, 0, 0)$, $(0, l_2, 0)$ and $(0, 0, l_3)$.

Now let N be the flat 2-torus obtained as the quotient of \mathbb{R}^2 by the subgroup generated by translations by $(l_1, 0)$ and $(0, l_3)$, and equip $F(N)$ with the Sasaki metric associated to the scalar l_2 . Then $F(M)$ and $F(N)$ are isometric but if l_1, l_2, l_3 are distinct, M and N are not isometric.

On the other hand if $l_1 = l_3 \neq l_2$, then this construction produces an isometry $F(M) \rightarrow F(N)$ that is not a bundle map.

Example 1.2 produces counterexamples to Question 1.1. Note that we used different bi-invariant metrics g_V on the fibers. Therefore to give a positive answer to Question 1.1 we must normalize the volume of the fibers of $F(M) \rightarrow M$.

Our main theorem is that under the assumption of normalization Question 1.1 has the following positive answer, except possibly in dimensions 3, 4 and 8.

Theorem A. *Let M, N be closed orientable connected Riemannian n -manifolds. Equip $F(M)$ and $F(N)$ with Sasaki metrics where the fibers of $F(M) \rightarrow M$ and $F(N) \rightarrow N$ have fixed volume $\lambda > 0$. Assume $n \neq 3, 4, 8$. Then M, N are isometric if and only if $F(M)$ and $F(N)$ are isometric.*

We do not know if counterexamples to Question 1.1 exist in dimensions 3, 4, and 8.

Outline of proof. Takagi-Yawata [TY94] give a description of the Lie algebra $i(X)$ of Killing fields of $X = F(M)$ except in dimensions 2, 3, 4 or 8, or when M has positive constant curvature. In the case of constant positive curvature $F(M)$ is isometric to $SO(n+1)/\pi_1(M)$, and we resolve this case in Section 4. If $n = 2$, then we finish the proof in Section 5 using the classification of surfaces and Lie groups in low dimensions. Since we assumed that $n \neq 3, 4$ or 8, we can then use the result of Takagi-Yawata.

Note that the Lie algebra $i(X)$ contains $\mathfrak{o}(n)$ acting transitively on the fibers of the natural bundle $\pi_M : X \rightarrow M$. Using the explicit computation of Takagi-Yawata we show that this is the only copy of $\mathfrak{o}(n)$ contained in $i(X)$, except in a few very special cases. In these cases we show that either $\text{Isom}(M)$ is extremely large or M is flat. We are able to resolve the flat case separately. In the case that $\text{Isom}(M)$ is very large we use classification theorems from the theory of compact transformation groups to prove that M and N are isometric.

Further, $i(X)$ also contains $\mathfrak{o}(n)$ acting transitively on the fibers of the bundle $\pi_N : X \rightarrow N$. Since we can assume that there is only one copy of $\mathfrak{o}(n)$ contained in $i(X)$, we see that the fibers of the bundles π_M and π_N coincide. We show that in this case M and N are isometric.

Outline of the paper. In Section 2 we will review preliminaries about actions of Lie groups of G on a manifold M when $\dim G$ is large compared to $\dim M$. In Section 3 we will prove the Main Theorem A except when M and N are surfaces or have metrics of constant positive curvature. The proof in the case that at least one of M or N has constant positive curvature will be given in Section 4. We prove Theorem A in the case that M and N are surfaces in Section 5.

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2. HIGH-DIMENSIONAL ISOMETRY GROUPS OF MANIFOLDS

In this section we review some known results about effective actions of a compact Lie group G on a closed n -manifold M when $\dim G$ is large compared to n . We will be especially interested in actions of $SO(n)$ on an n -manifold M . First, there is the following classical upper bound for the dimension of a compact group acting smoothly on an n -manifold.

Theorem 2.1 ([Kob72, II.3.1]). *Let M be a closed n -manifold and G a compact group acting smoothly and effectively on M . Then $\dim G \leq \frac{n(n+1)}{2}$. Further equality holds if and only if M is isometric to either S^n or $\mathbb{R}P^n$ with a metric of constant positive curvature. In this case we $G = PSO(n)$ or $SO(n)$ or $O(n)$, and G acts on M in the standard way.*

This leads us to study groups of dimension $< \frac{n(n+1)}{2}$. First, there is the following remarkable gap theorem due to H.C. Wang.

Theorem 2.2 (H.C. Wang [Wan47]). *Let M be a closed n -manifold with $n \neq 4$. Then there is no compact group G acting effectively on M with*

$$\frac{n(n-1)}{2} + 1 < \dim G < \frac{n(n+1)}{2}.$$

Therefore the next case to consider is $\dim G = \frac{n(n-1)}{2} + 1$. The following characterization is independently due to Kuiper and Obata.

Theorem 2.3 (Kuiper, Obata [Kob72, II.3.3]). *Let M be a closed Riemannian n -manifold with $n > 4$ and G a connected compact group of dimension $\frac{n(n-1)}{2} + 1$ acting smoothly and effectively on M . Then M is isometric to $S^{n-1} \times S^1$ or $\mathbb{R}P^{n-1} \times S^1$ equipped with a product of a round metric on S^{n-1} or $\mathbb{R}P^{n-1}$ and the standard metric on S^1 . Further $G = SO(n) \times S^1$ or $PSO(n) \times S^1$.*

After Theorem 2.3, the natural next case to consider is $\dim G = \frac{n(n-1)}{2}$. There is a complete classification due to Kobayashi-Nagano [KN72].

Theorem 2.4 (Kobayashi-Nagano). *Let M be a closed Riemannian n -manifold with $n > 5$ and G a connected compact group of dimension $\frac{n(n-1)}{2}$ acting smoothly and effectively on M . Then M must be one of the following.*

- (1) *M is diffeomorphic to S^n or $\mathbb{R}P^n$ and $G = SO(n)$. In this case G has a fixed point on M . Every orbit is either a fixed point or has codimension 1. Regarding S^n as the solution set of $\sum_{i=0}^n x_i^2 = 1$ in \mathbb{R}^{n+1} , the metric on M (or its double cover if M is diffeomorphic to $\mathbb{R}P^n$) is of the form*

$$ds^2 = f(x_0) \sum_{i=0}^n dx_i^2$$

for a smooth positive function f on $[-1, 1]$.

- (2) *M is a fiber bundle $L_M \rightarrow M \rightarrow S^1$ where L_M is either S^{n-1} or $\mathbb{R}P^{n-1}$. In this case G acts on M preserving each fiber, and the action on each fiber is by an orthogonal group.*

(3) M is a quotient $(S^{n-1} \times \mathbb{R})/\Gamma$ where Γ is generated by

$$(v, t) \mapsto (v, t + 2)$$

$$(v, t) \mapsto (-v, -t).$$

In this case $G = SO(n)$ acting orthogonally on the image of each copy $S^{n-1} \times \{t\}$ in M . We have $M/G = [0, 1]$. The G -orbits lying over the endpoints $0, 1$ are isometric to round projective spaces $\mathbb{R}P^{n-1}$ and the G -orbits lying over points in $(0, 1)$ are round spheres.

(4) If $n = 6$ there is the additional case that M is a simply-connected Kähler manifold of complex dimension 3 with constant holomorphic sectional curvature, and G is the largest connected group of holomorphic isometries.

(5) If $n = 7$ there is the additional case $M \cong Spin(7)/G_2$ and $G = Spin(7)$. In this case M is isometric to S^7 with a constant curvature metric.

Remark 2.5. Actually Kobayashi-Nagano prove a more general result that includes the possibility that M is noncompact, and there are more possibilities. Since we will not need the noncompact case, we have omitted these. Specializing to the compact case gives an explicit description of Case 4 as follows. Hawley [Haw53] and Igusa [Igu54] independently proved that a simply-connected complex n -manifold of constant holomorphic sectional curvature is isometric to either $\mathbb{C}^n, \mathbb{B}^n$ or $\mathbb{C}P^n$ (with standard metrics). Therefore in Case (4) we obtain that M is isometric to $\mathbb{C}P^3$ (equipped with a scalar multiple of the Fubini-Study metric) and $G = SO(6) \cong SU(4)/\{\pm \text{id}\}$.

Theorem 2.4 does not cover the case $n = 5$. Under the additional assumption that $G = SO(5)$ we resolve this case in the following proposition.

Proposition 2.6. *Let M be a closed Riemannian 5-manifold and suppose $G = SO(5)$ acts on M smoothly and effectively. Then M admits a description as in Cases (1), (2) or (3) of Theorem 2.4.*

Proof. The proof of Theorem 2.4 (see [KN72, Section 3]) shows that the assumption that $n > 5$ is only used to show that no G -orbit has codimension 2. We will show that if $G = SO(5)$ and $n = 5$, then there are still no codimension 2 orbits, so that the rest of the proof of Theorem 2.4 applies.

So suppose that $x \in M$ and that the orbit $G(x)$ has codimension 2 in M . Let G_x be the stabilizer of x . Then we see

$$\dim G_x = \dim G - \dim G(x) = \frac{(n-1)(n-2)}{2} + 1.$$

Now we apply the following lemma Montgomery-Samelson [MS43] that characterizes high-dimensional subgroups of orthogonal groups.

Lemma 2.7 (Montgomery-Samelson). *$O(n)$ contains no proper closed subgroup of dimension $> \frac{1}{2}(n-1)(n-2)$ other than $SO(n)$ unless $n = 4$.*

This is a contradiction. □

3. PROOF OF THEOREM A

Before starting the proof of Theorem A we will record the following observation about manifolds with isometric frame bundles.

Lemma 3.1. *Fix $\lambda > 0$. Let M, N be closed orientable connected Riemannian n -manifolds and equip $F(M)$ and $F(N)$ with Sasaki metrics where the fibers of $F(M) \rightarrow M$ and $F(N) \rightarrow N$ have volume λ . Suppose that $F(M)$ and $F(N)$ are isometric. Then $\text{vol}(M) = \text{vol}(N)$.*

Proof. Set $X := F(M) \cong F(N)$. Since the fiber bundle $X \rightarrow M$ has fibers with volume λ , we have

$$\text{vol}(X) = \frac{\text{vol}(M)}{\lambda}.$$

Likewise we have $\text{vol}(X) = \frac{\text{vol}(N)}{\lambda}$. Combining these we get $\text{vol}(M) = \text{vol}(N)$. \square

We prove Theorem A.

Proof. Write $X := F(M) \cong F(N)$, and let

$$\begin{aligned}\pi_M : X &\rightarrow M \\ \pi_N : X &\rightarrow N\end{aligned}$$

be the natural projections. $SO(n)$ acts transitively and freely on each of the fibers of π_M and π_N . Identifying a fiber with $SO(n)$ under this action, the metric on the fiber over x is given by a bi-invariant metric on $SO(n)$. On $\mathfrak{o}(n)$ such a metric can be written as

$$\langle A, B \rangle = \mu_x \sum_{i,j} A_{ij} B_{ij},$$

where $A, B \in \mathfrak{o}(n)$ and $\mu_x > 0$ is some scalar. In fact μ_x does not depend on x since the volume of all fibers is equal to the fixed constant λ . Therefore by rescaling the metrics on M and N we may assume that $\mu_x = 1$. A theorem by Takagi-Yawata [TY94] computes the Lie algebra $i(X)$ of Killing fields on X as

$$i(X) = ((\Lambda^2 M)_0 \rtimes i(M)) \oplus i_V^M \quad (3.1)$$

unless either $n = 2, 3, 4$ or 8 , or M has positive constant curvature $\frac{1}{2}$. Here $i(M)$ is the Lie algebra of Killing fields on M (which are naturally lifted to $F(M)$), $i_V^M \cong \mathfrak{o}(n)$ is the natural action of $\mathfrak{o}(n)$ on fibers of the principal bundle π_M , and $(\Lambda^2 M)_0$ is the Lie algebra of parallel forms on M (these naturally induce Killing fields on X , see [TY91]). By assumption we have $n \neq 3, 4$, or 8 . The proof of Theorem A if M has positive constant curvature $\frac{1}{2}$ is given in Section 4, and if $n = 2$ we give the proof in Section 5. Henceforth we assume that $n \neq 2$ and M does not have positive constant curvature $\frac{1}{2}$, so that the decomposition of Equation 3.1 holds.

The natural action of $SO(n)$ on the fibers of π_N induces an embedding of $SO(n)$ in $\text{Isom}(X)$, hence an embedding of Lie algebras

$$i_V^N \hookrightarrow i(X) = ((\Lambda^2 M)_0 \rtimes i(M)) \oplus i_V^M.$$

We identify i_V^N with its image throughout. Now consider the images of the projections of i_V^N onto $(\Lambda^2 M)_0 \rtimes i(M)$ and i_V^M . We have the following cases:

- (1) i_V^N projects trivially to $(\Lambda^2 M)_0 \rtimes i(M)$. Then $i_V^N \subseteq i_V^M$. Since we also know $\dim i_V^N = \dim i_V^M$, we have $i_V^N = i_V^M$.
- (2) i_V^N projects nontrivially to $(\Lambda^2 M)_0 \rtimes i(M)$, but the further projection of i_V^N to $i(M)$ is trivial. Then the image of i_V^N in $(\Lambda^2 M)_0 \rtimes i(M)$ is contained in $(\Lambda^2 M)_0$. Further since $i_V^N \cong \mathfrak{o}(n)$ is simple, we must have $\mathfrak{o}(n) \subseteq (\Lambda^2 M)_0$.
- (3) i_V^N projects nontrivially to $(\Lambda^2 M)_0 \rtimes i(M)$ and the further projection of i_V^N to $i(M)$ is also nontrivial.

We consider each of these cases separately.

Case 1 (vertical directions agree). Assume that $i_V^N = i_V^M$. Since the values of $i_V^M(X)$ at any point $x \in X$ span the vertical tangent space $T_x^V X$, it follows that the fibers of π_M and π_N actually coincide. Hence we have a natural map $f : M \rightarrow N$.

We claim f is an isometry. Denote by $T^H X$ and $T^V X$ the horizontal and vertical subbundles with respect to $\pi_M : X \rightarrow M$. Because π_M is a Riemannian submersion, the metric on $T_x M$ coincides with the metric on the horizontal subbundle $T_u^H X$ at a point $u \in \pi_M^{-1}(x)$. We have

$$T_u^H X = (T_u^V X)^\perp = (\ker(\pi_M)_*)^\perp = (\ker(\pi_N)_*)^\perp.$$

The latter is the horizontal subbundle with respect to π_N . Since π_N is a Riemannian submersion, the metric on $T_u^H X$ coincides with the metric on $T_{\pi_N(u)} X$. This proves the claim. This proves the naturally induced map

$$f : M \rightarrow N$$

is a local isometry. It is also injective, so M and N are isometric.

Case 2 (many parallel forms). Assume that $\mathfrak{o}(n) \subseteq (\Lambda^2 M)_0$. We claim that M is isometric to a flat manifold. Note that

$$\dim(\Lambda^2 M)_0 \geq \dim \mathfrak{o}(n) = \frac{n(n-1)}{2}.$$

On the other hand, since a parallel form is invariant under parallel transport, it is determined by its values on a single tangent space, hence we have an embedding

$$(\Lambda^2 M)_0 \hookrightarrow \Lambda^2 T_x M.$$

Therefore $\dim(\Lambda^2 M)_0 \leq \frac{n(n-1)}{2}$. So we have $\mathfrak{o}(n) \cong (\Lambda^2 M)_0$. A parallel 2-form ω on M lifts to a parallel 2-form $\tilde{\omega}$ on \tilde{M} , and $\tilde{\omega}$ is invariant under the holonomy group. Suppose now that $\tilde{T} : T_{\tilde{x}} \tilde{M} \rightarrow T_{\tilde{x}} \tilde{M}$ is nontrivial and belongs to the holonomy group. The covering map $\tilde{M} \rightarrow M$ induces an isomorphism $T_{\tilde{x}} \tilde{M} \cong T_x M$. Write $T : T_x M \rightarrow T_x M$ for the map induced by \tilde{T} under this identification. Since \tilde{T} is nontrivial we can choose $\omega \in (\Lambda^2 M)_0$ such that $\omega|_x$ is not fixed by T . This is a contradiction since ω is parallel.

So \tilde{M} has trivial holonomy. Since the holonomy algebra contains the algebra generated by curvature operators $R(v, w)$, it follows that $R(v, w) = 0$ for all $v, w \in T_x M$, i.e. \tilde{M} is flat. We conclude that M is a closed flat manifold. Therefore $i(M)$ is abelian, and recall that we have

$$i(X) \cong i_V^M \oplus ((\Lambda^2 M)_0 \rtimes i(M))$$

We know that $i_V^N \cong \mathfrak{o}(n)$ has no abelian quotients, so $i_V^N \subseteq i_V^M \oplus (\Lambda^2 M)_0$. Since the vector fields in $i_V^M \oplus (\Lambda^2 M)_0$ are vertical with respect to π_M , it follows that for $x \in N$ and $\tilde{x} \in \pi_N^{-1}(x)$, we have

$$T_{\tilde{x}} \pi_N^{-1}(x) = i_V^N|_{\tilde{x}} \subseteq (i_V^M \oplus (\Lambda^2 M)_0)|_{\tilde{x}} = T_{\tilde{x}} \pi_M^{-1}(\pi_M(\tilde{x})).$$

Since $\pi_N^{-1}(x)$ and $\pi_M^{-1}(\pi_M(\tilde{x}))$ are connected submanifolds with the same dimension, we have $\pi_N^{-1}(x) = \pi_M^{-1}(\pi_M(\tilde{x}))$. Therefore the fibers of π_M and π_N agree. We conclude that M and N are isometric in the same way as Case 1.

Case 3 (many Killing fields). Assume i_V^N projects nontrivially to $i(M)$. Since $n > 4$, we know that $\mathfrak{o}(n)$ is simple. By assumption $i_V^N \cong \mathfrak{o}(n)$ projects nontrivially to $i(M)$, hence i_V^N projects isomorphically to $i(M)$. Let \mathfrak{h} be the image of i_V^N in $i(M)$. We first claim the following symmetry in the situation for M and N .

Claim 3.2. Assume that $\mathfrak{o}(n) \not\subseteq (\Lambda^2 M)_0$ and that $\mathfrak{o}(n) \not\subseteq (\Lambda^2 N)_0$. Then

- (1) $i_V^N \subseteq i(M)$, and
- (2) $i_V^M \subseteq i(N)$.

Proof. Note that i_V^M and \mathfrak{h} centralize each other and are isomorphic to $\mathfrak{o}(n)$. Consider the projection

$$p_1 : \mathfrak{h} \oplus i_V^M \subseteq i(X) \cong i_V^N \oplus ((\Lambda^2 N)_0 \rtimes i(N)) \rightarrow i_V^N.$$

Note that

$$\begin{aligned} \dim(\mathfrak{h} \oplus i_V^M) &= (n-1)(n-2) \\ &> \frac{n(n-1)}{2} = \dim i_V^N \end{aligned}$$

since $n > 4$. Therefore p_1 cannot be injective. If p_1 is trivial, then we have

$$\mathfrak{h} \oplus i_V^M \subseteq (\Lambda^2 N)_0 \rtimes i(N).$$

Using that $\mathfrak{o}(n)$ is simple, and since $(\Lambda^2 N)_0$ does not contain a copy of $\mathfrak{o}(n)$ by assumption, we must have that $\mathfrak{h} \oplus i_V^M$ projects isomorphically to $i(N)$. However note that $\dim i(N) \leq \frac{n(n+1)}{2}$ by Theorem 2.1. Again comparing dimensions we see that this is impossible. Therefore $\ker p_1$ is a proper ideal of $\mathfrak{h} \oplus i_V^M$, so $\ker p_1$ is either \mathfrak{h} or i_V^M .

Now consider the projection

$$p_2 : \mathfrak{h} \oplus i_V^M \subseteq i(X) \cong i_V^N \oplus ((\Lambda^2 N)_0 \rtimes i(N)) \rightarrow (\Lambda^2 N)_0 \rtimes i(N).$$

Because $(\Lambda^2 N)_0$ does not contain a copy of $\mathfrak{o}(n)$, we see that $p_2(\mathfrak{h} \oplus i_V^M)$ projects isomorphically to $i(N)$. As above we see that p_2 can be neither injective nor trivial. Hence we also have that $\ker p_2$ is either \mathfrak{h} or i_V^M .

If $\ker p_2 = i_V^M$, then we have $i_V^M = i_V^N$, but this contradicts the assumption that i_V^N projects nontrivially to $i(M)$. Therefore we must have $\ker p_1 = i_V^M$ and $\ker p_2 = \mathfrak{h}$. The latter implies $i_V^N = \mathfrak{h}$, which proves (1).

Since $\ker p_1 = i_V^M$, we have $i_V^M \subseteq (\Lambda^2 N)_0 \rtimes i(N)$ and i_V^M projects isomorphically into $i(N)$. This allows us to repeat the entire preceding argument that proved (1) with M and N switched, which proves (2). \square

If $\mathfrak{o}(n) \subseteq (\Lambda^2 M)_0$ or $\mathfrak{o}(n) \subseteq (\Lambda^2 N)_0$, the proof is finished in Case 2. Therefore we assume $i_V^N \subseteq i(M)$ and $i_V^M \subseteq i(N)$. Write $H_M := \exp(i_V^N)$ where \exp is with respect to the Lie group $\text{Isom}(M)$. Similarly, write $H_N := \exp(i_V^M)$ where \exp is with respect to $\text{Isom}(N)$. Since $i_V^N \cong i_V^M \cong \mathfrak{o}(n)$, we can apply the results of Section 2 in this case.

Case 3(a) (H_M or H_N acts transitively). Suppose H_M acts transitively on M . By Theorem 2.4, Remark 2.5 and Proposition 2.6, we know that M is isometric to \mathbb{CP}^3 or S^7 . Since we assumed that M does not have positive constant curvature, we must have $M \cong \mathbb{CP}^3$. Now consider the action of H_N on N . From the classification in Theorem 2.4, we see that N must be one of the following:

- (1) diffeomorphic to S^6 or \mathbb{RP}^6 ,
- (2) a fiber bundle $L_N \rightarrow N \rightarrow S^1$ where L_N is S^5 or \mathbb{RP}^5 , or
- (3) isometric to \mathbb{CP}^3 with a metric of constant holomorphic sectional curvature.

Since $F(M) = F(\mathbb{CP}^3)$ and $SO(6)$ have finite fundamental groups, we see that Case (2) is impossible. The long exact sequence on homotopy groups of the fibration $SO(6) \rightarrow F(\mathbb{CP}^3) \rightarrow \mathbb{CP}^3$ gives

$$1 = \pi_2 SO(6) \rightarrow \pi_2(F(\mathbb{CP}^3)) \rightarrow \pi_2(\mathbb{CP}^3) \rightarrow \pi_1(SO(6)) = \mathbb{Z}/(2\mathbb{Z}).$$

Since $\pi_2(\mathbb{CP}^3) \cong \mathbb{Z}$ it follows that $\pi_2(F(\mathbb{CP}^3)) \cong \mathbb{Z}$. On the other hand we have $\pi_2(F(S^6)) = \pi_2(SO(7)) = 1$ and similarly $\pi_2(F(\mathbb{RP}^6)) = 1$. Therefore Case (1) is impossible as well. We conclude that M and N are both isometric to \mathbb{CP}^3 with a metric of constant holomorphic sectional curvature.

A metric of constant holomorphic sectional curvature on \mathbb{CP}^3 is determined by a bi-invariant metric on $SU(4)$, which is then induced on the quotient $SU(4)/U(3) \cong \mathbb{CP}^3$.

Hence the metrics on M and N differ only by scaling, so M and N are isometric if and only if $\text{vol}(M) = \text{vol}(N)$. By Lemma 3.1 we have $\text{vol}(M) = \text{vol}(N)$ so M and N are indeed isometric.

Case 3(b) (H_M and H_N do not act transitively). Theorem 2.4 and Proposition 2.6 imply that M and N are of one of the following types:

- (1) diffeomorphic to S^n or $\mathbb{R}P^n$,
- (2) a fiber bundle $F \rightarrow E \rightarrow S^1$ where each fiber is isometric to a round sphere or projective space, or
- (3) $(S^{n-1} \times \mathbb{R})/\Gamma$ where $\Gamma \cong D_\infty$ is generated by $(v, t) \mapsto (v, t + 2)$ and $(v, t) \mapsto (-v, -t)$.

Claim 3.3. M and N belong to the same types in the above classification.

Proof. The fiber bundles $X \rightarrow M$ and $X \rightarrow N$ give long exact sequences on homotopy groups

$$\pi_2(M) \rightarrow \pi_1(SO(n)) \rightarrow \pi_1(X) \rightarrow \pi_1(M) \rightarrow 1$$

and likewise for N . Since $\pi_2(M) = \pi_2(N) = 1$ in all the above cases, we have a short exact sequence

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1(X) \rightarrow \pi_1(M) \rightarrow 1$$

and likewise for N . We see that $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$ precisely when M is diffeomorphic to S^n , and $\pi_1(X)$ has order 4 precisely when M is diffeomorphic to $\mathbb{R}P^n$. If $\pi_1(X)$ is infinite then M is of type (2) or (3). If the maximal finite subgroup of $\pi_1(X)$ has order 2 then M is of type (2), and if the maximal finite subgroup of $\pi_1(X)$ has order 4 then M is of type (3). Therefore we can distinguish all the possible cases by considering $\pi_1(X)$. It follows that M and N are of the same type. \square

Case A (M and N are of type (2)). We obtain more information by using that H_M acts on the frame bundle $F(M)$ by flows of the Killing fields i_V^N as follows. We claim that M is isometric to $S^{n-1} \times S^1$ or $\mathbb{R}P^{n-1} \times S^1$ equipped with a product metric, and the metric on the spheres $S^{n-1} \times \{z\}$ is round with radius r where r only depends on λ . Since in addition $\text{vol}(M) = \text{vol}(N)$ by Lemma 3.1, it will then follow that the S^1 -factors of M and N are isometric, and hence that M and N are isometric.

So let us prove that M is isometric to a product. Write M as a fiber bundle

$$L_M \rightarrow M \xrightarrow{q_M} S^1$$

where all fibers L_M are isometric to round spheres or projective spaces. Consider the bundle

$$F(L_M) \rightarrow F_1(M) \rightarrow S^1 \tag{3.2}$$

where the fiber over a point $z \in M$ is the frame bundle $F(q_M^{-1}(z))$. Define a map

$$i : F_1(M) \hookrightarrow F(M)$$

in the following way. A point $x \in F_1(M)$ consists of a frame at a point $p \in L$. Since the foliation by fibers of the fiber bundle 3.2 is transversely oriented, x can be extended to a frame for M at p by adding to x the unique unit vector $v \in T_p M$ such that (x, v) is an oriented orthonormal frame for M . Then $i(F_1(M))$ is an I_V^N -invariant submanifold of $F(M)$. Since the orbits of I_V^N in $F(M)$ are totally geodesic, the foliation \mathcal{F} by I_V^N -orbits on $F_1(M)$ is a totally geodesic codimension 1 foliation. Consider the horizontal foliation $\mathcal{H} := \mathcal{F}^\perp$ of $F_1(M)$. Since \mathcal{H} is 1-dimensional, it is integrable.

Johnson-Whitt proved that if the horizontal distribution associated to a totally geodesic foliation is integrable, then the horizontal distribution is also totally geodesic [JW80, Theorem 1.6]. Further they showed that a manifold with two orthogonal totally geodesic foliations is locally a Riemannian product [JW80, Proposition 1.3].

Therefore $F_1(M)$ is locally a Riemannian product $F \times H$ where F (resp. H) is an open neighborhood in a leaf of \mathcal{F} (resp. \mathcal{H}). Since the fibers of the projection $p : F_1(M) \rightarrow M$ are contained in the leaves of \mathcal{F} , it follows that $p(F \times H) = p(F) \times p(H)$. Suppose that $(x, y) \in p(F) \times p(H)$ and choose lifts $\tilde{x} \in p^{-1}(x)$ of x and $\tilde{y} \in p^{-1}(y)$ of y . Since p is a Riemannian submersion, we see that M is a local Riemannian product $p(F) \times p(H)$. In particular there exists a unique unit length Killing field Z on M such that Z is orthogonal to the leaves of $p_*\mathcal{F}$. Since each leaf of $p_*\mathcal{F}$ consists of a single H_M -orbit, Z is orthogonal to H_M -orbits. Hence we have the following lower bound on the dimension of the Lie algebra of Killing fields $i(M)$

$$\dim i(M) \geq \dim H_M + 1 = \frac{1}{2}n(n-1) + 1.$$

By Theorem 2.3 we know that M is isometric to a product $L \times S^1$. This proves Case A.

Case B (M and N are of type (3)). The unique torsion-free, index 2 subgroups of $\pi_1(M)$ and $\pi_1(N)$ give double covers M' and N' . We claim that the frame bundles $F(M')$ and $F(N')$ are also isometric. The fiber bundle $SO(n) \rightarrow X \rightarrow M$ gives

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1(X) \rightarrow D_\infty \rightarrow 1.$$

Now $\pi_1(F(M'))$ and $\pi_1(F(N'))$ are both index 2 subgroups of $\pi_1(X)$. Since M' and N' are diffeomorphic to $S^{n-1} \times S^1$ we see that $\pi_1(F(M')) \cong (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$ and likewise for $\pi_1(F(N'))$. Therefore $\pi_1(F(M'))$ and $\pi_1(F(N'))$ correspond to the same index 2 subgroup of $\pi_1(X)$. It follows that $F(M')$ and $F(N')$ are also isometric.

Since M' and N' are diffeomorphic to $S^{n-1} \times S^1$ and H_M acts on S^{n-1} orthogonally, the argument from Case A applies and yields that M' and N' are isometric to the same product $S^{n-1} \times S^1$. Then M and N are obtained as the quotient of $S^{n-1} \times S^1$ by the map $(v, z) \mapsto (-v, z^{-1})$. Hence M and N are isometric.

Case C (M and N are of type (1)). Suppose H_M acts on M with a fixed point. By Theorem 2.4 and Proposition 2.6 we know that M is diffeomorphic to a standard sphere or projective space. Further the metric on M (or its double cover if M is diffeomorphic to $\mathbb{R}P^n$) is of the form

$$ds_M^2 = f_M(x_0) \sum_{i=0}^n dx_i^2 \quad (3.3)$$

where we view S^n as the locus $\sum_{i=0}^n x_i^2 = 1$ in \mathbb{R}^{n+1} . Similarly the metric on N can be written as

$$ds_N^2 = f_N(x_0) \sum_{i=0}^n dx_i^2 \quad (3.4)$$

Identify

$$N/I_V^M = X/(I_V^M I_V^N) = M/I_V^M = [-1, 1].$$

Let $-1 < x < 1$ and choose a lift $y_M \in M$ of x . Equation 3.3 shows that $\text{vol}(H_M y_M) = f_M(x) \text{vol}(S^{n-1})$ where S^{n-1} is equipped with the metric $\sum_{i=1}^n dx_i^2$. Similarly if y_N is a lift of x to N we have $\text{vol}(H_N y_N) = f_N(x) \text{vol}(S^{n-1})$. Now choose a common lift \tilde{y} of y_M and y_N to X . On the one hand we have

$$\text{vol}(I_V^M I_V^N \tilde{y}) = \lambda \text{vol}(H_M y_M) = \lambda f_M(x) \text{vol}(S^{n-1})$$

and on the other hand we have

$$\text{vol}(I_V^M I_V^N \tilde{y}) = \lambda \text{vol}(H_N y_N) = \lambda f_N(x) \text{vol}(S^{n-1}).$$

It follows that $f_M(x) = f_N(x)$. Hence M and N are isometric.

4. PROOF FOR M WITH POSITIVE CONSTANT CURVATURE $\frac{1}{2}$

Assume that we normalized the metric on the fibers as in Section 3, so that for $A, B \in \mathfrak{o}(n) \cong T_x^V M$ we have

$$\langle A, B \rangle = \sum_{i,j} A_{ij} B_{ij}.$$

Assume further that M has positive constant curvature $\frac{1}{2}$. We have

$$M \cong S^n / \pi_1(M) \cong SO(n) \backslash SO(n+1) / \pi_1(M),$$

where $SO(n) \subseteq SO(n+1)$ is the standard embedded copy. Hence we have $X \cong SO(n+1) / \pi_1(M)$, where the cover $SO(n+1)$ is equipped with a bi-invariant metric. Similarly we have that N is isometric to $L \backslash SO(n+1) / \pi_1(N)$, where $L \cong SO(n)$ acts on $SO(n+1)$ isometrically.

A result of d'Atri-Ziller [DZ79] computes the isometry group of a simple compact Lie group G equipped with a bi-invariant metric, which yields

$$\text{Isom}(G) \cong G \rtimes \text{Aut}(G)$$

where the copy of G acts by left-translations. Since $\text{Out}(G)$ is discrete and L is connected, it follows that L is contained in the group of left- and right-translations, so we have an embedding

$$L \hookrightarrow SO(n+1) \times SO(n+1).$$

We claim that L is contained in one factor. To see this, assume the contrary. Since L is simple, L projects isomorphically onto each factor, hence can be realized as the graph of an injective homomorphism

$$\varphi : SO(n) \hookrightarrow SO(n+1).$$

Note that φ is obtained as conjugation of a standard copy of $SO(n)$ by an element $h \in SO(n+1)$. Further since

$$\dim N = \dim SO(n+1) - \dim L,$$

every stabilizer of the action of L on $SO(n+1)$ is finite. However, we can compute that h^{-1} is a fixed point for the L -action as follows. Every element of L is of the form (g, hgh^{-1}) for some $g \in SO(n)$, and

$$(g, hgh^{-1}) \cdot h^{-1} = gh^{-1}(hgh^{-1})^{-1} = h^{-1}.$$

This is a contradiction. It follows that L consists of either left- or right-translations.

Again we can conjugate L to a standard copy of $SO(n)$ by an element of $SO(n+1)$. Therefore without loss of generality we have $N \cong SO(n) \backslash SO(n+1) / \pi_1(N)$, and we have an isometry

$$f : \text{Spin}(n+1) / \pi_1(M) \cong F(M) \rightarrow F(N) \cong \text{Spin}(n+1) / \pi_1(N).$$

Here $\text{Spin}(n) \rightarrow SO(n)$ is the universal cover of $SO(n)$. By composing with a left-translation of $SO(n+1)$, we can assume $f(e\pi_1(M)) = e\pi_1(N)$. Lift f to an isometry

$$\tilde{f} : \text{Spin}(n+1) \rightarrow \text{Spin}(n+1).$$

We can assume that $\tilde{f}(e) = e$ by choosing an appropriate lift. Hence by the computation of $\text{Isom}(\text{Spin}(n+1))$ by d'Atri-Ziller, \tilde{f} is an automorphism. Again by composing \tilde{f} with conjugation by an element in $\text{Spin}(n+1)$, we can assume that $\tilde{f}(\text{Spin}(n)) = \text{Spin}(n)$, and hence \tilde{f} descends to an isometry

$$\bar{f} : S^n \rightarrow S^n,$$

where we identified S^n with $\text{Spin}(n) \backslash \text{Spin}(n+1)$. Since \tilde{f} conjugates $\pi_1(M)$ to $\pi_1(N)$, we further know that \bar{f} descends to an isometry $M \rightarrow N$. \square

5. PROOF OF THE MAIN THEOREM FOR SURFACES

In this section we prove Theorem A for surfaces. We cannot use the Takagi-Yawata theorem that computes $i(X)$ in this situation, but instead we use the classification of surfaces and Lie groups in low dimensions.

Proof. Let M and N be closed oriented surfaces with $F(M) \cong F(N)$. Therefore M and N are each diffeomorphic to one of S^2 , T^2 or Σ_g with $g \geq 2$. We know that

- $F(S^2)$ is diffeomorphic to $SO(3)$,
- $F(T^2)$ is diffeomorphic to T^3 , and
- $F(\Sigma_g)$ is diffeomorphic to $T^1\Sigma_g = \mathrm{PSL}_2\mathbb{R}/\Gamma$ for a cocompact torsion-free lattice $\Gamma \subseteq \mathrm{PSL}_2\mathbb{R}$.

If $i_V^M = i_V^N$, then we proceed as in Case 1 in the proof of Theorem A, and we find that M and N are isometric. Therefore we will assume that $\dim i(X) \geq 2$.

Case 1 (M and N are diffeomorphic to $\Sigma_g, g \geq 2$). Then $X = T^1\Sigma_g$ is a closed aspherical manifold. Conner and Raymond proved [CR70] that if a compact connected Lie group G acts effectively on a closed aspherical manifold L , then G is a torus and $\dim G \leq \mathrm{rk}_{\mathbb{Z}} Z(\pi_1 L)$. In particular we find that $\dim i(X) \leq \mathrm{rk}_{\mathbb{Z}} Z(\pi_1 T^1\Sigma_g) = 1$, which is a contradiction.

Case 2 (M and N are diffeomorphic to S^2). Write $G := \mathrm{Isom}(X)^0$. If $\dim G = 2$, then G is a 2-torus. In particular I_V^M and I_V^N centralize each other. Therefore I_V^M acts on $X/I_V^N = N$ and similarly I_V^N acts on M . Since an S^1 -action on S^2 has at least one fixed point (because $\chi(S^2) \neq 0$), we see that $N/I_V^M \cong [-1, 1] \cong M/I_V^N$. It is then straightforward to see that the metrics on M (resp. N) is of the form

$$ds_M^2 = f_M(x_0)(dx_0^2 + dx_1^2)$$

(resp. $ds_N^2 = f_N(x_0)(dx_0^2 + dx_1^2)$) as in Theorem 2.4.(1). We can apply the reasoning from Case C of the proof of Case 3(b) of Theorem A to show M and N are isometric.

Therefore we will assume $\dim G^0 \geq 3$. The centralizer $C_G(I_V^M)$ of I_V^M acts on M with kernel I_V^M , and M is diffeomorphic to S^2 . In particular $C_G(I_V^M)/I_V^M$ has rank 1, because T^2 does not act effectively on S^2 . To see this, note that any 1-parameter subgroup H of T^2 has a fixed point on S^2 (since any vector field has a zero on S^2). We can take H to be dense in T^2 , so that T^2 fixes a point p . Therefore T^2 embeds in $SO(T_p M) \cong SO(2)$, which is a contradiction.

In addition we know that $\dim G \leq 6$ by Theorem 2.1. So the only possibilities for G are

- (a) $\mathfrak{g} \cong o(3)$,
- (b) $\mathfrak{g} \cong \mathbb{R} \oplus o(3)$, and
- (c) $\mathfrak{g} \cong o(3) \oplus o(3)$.

Case 2(a) ($\mathfrak{g} \cong o(3)$). Since G has rank 1, I_V^M and I_V^N are both maximal tori of G . Since all maximal tori are conjugate, there is some element $g \in G$ so that $gI_V^N g^{-1} = I_V^M$. Then g induces an obvious isometry $M \rightarrow N$.

Case 2(b) ($\mathfrak{g} \cong \mathbb{R} \oplus o(3)$). Since G has factors of rank ≥ 2 , we can conjugate I_V^M by an element $g \in G$ so that $gI_V^M g^{-1}$ and I_V^N centralize each other. As above (in the case that $G = T^2$ in Case 2) we see that M and N are isometric by applying the argument of Case C of Case 3(b) of the proof of Theorem A.

Case 2(c) ($\mathfrak{g} \cong o(3) \oplus o(3)$). In this case $\dim \mathrm{Isom}(X) = 6$ is maximal. By Theorem 2.1 the metric on X has positive constant curvature. Therefore the metrics on M and N have positive constant curvature. Further by Lemma 3.1 we have $\mathrm{vol}(M) = \mathrm{vol}(N)$. It follows that M and N are isometric.

Case 3 (M and N are diffeomorphic to T^2). In this case X is diffeomorphic to T^3 . Again by the theorem of Conner-Raymond [CR70] on actions of compact Lie groups on aspherical manifolds, we know that a connected compact Lie group acting on a torus is a torus. Therefore I_V^M and I_V^N centralize each other, so that I_V^N acts on $M = X/I_V^M$. Again by [CR70], the action of I_V^N on M is free, so that the map

$$M \rightarrow M/I_V^N \cong S^1$$

is a fiber bundle (with S^1 fibers). The argument of Case A in Case 3(b) of the proof of Theorem A constructs a (unit length) Killing field X_M on M that is orthogonal to the fibers of $M \rightarrow M/I_V^N$. Therefore M is in fact flat. Similarly we construct a unit length Killing field X_N on N that is orthogonal to the fibers of $N \rightarrow N/I_V^N$. Hence we conclude that N is flat.

A flat torus is specified by the length of two orthogonal curves that generate its fundamental group. For M we can consider the curves given by an I_V^N -orbit on M and an integral curve of X_M . Similarly for N we can consider an I_V^N -orbit on N and an integral curve of X_N .

For $x \in M$ and $\tilde{x} \in X$ lying over x , we have a covering

$$I_V^N \tilde{x} \rightarrow I_V^N x$$

of degree $|I_V^M \cap I_V^N|$. Therefore

$$\ell(I_V^N x) = \frac{1}{|I_V^M \cap I_V^N|} \ell(I_V^N \tilde{x}) = \frac{\lambda}{|I_V^M \cap I_V^N|}.$$

Combining this with a similar computation for the length of an I_V^M -orbit on N gives $\ell(I_V^N x) = \ell(I_V^M y)$ for every $x \in M$ and $y \in N$. Therefore we see that the length of an integral curve of X_M (resp. X_N) is $\frac{\text{vol}(M)}{\ell(I_V^N \cdot x)}$ for $x \in M$ (resp. $\frac{\text{vol}(N)}{\ell(I_V^M \cdot y)}$ for $y \in N$). Since $\text{vol}(M) = \text{vol}(N)$ by Lemma 3.1, it follows that M and N are isometric. \square

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